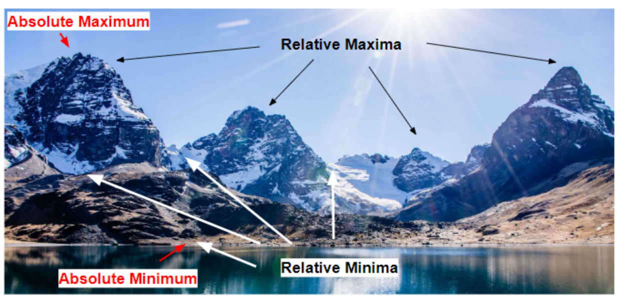
Math 2450: Extrema of Functions (Maxima and Minima)

What are extrema? We have all heard of the terms maximum and minimum before, where maximum is the greatest possible value and minimum is the smallest possible value. These are "extreme" values are known as **extrema**. We are going to explore different types of extrema of functions of two variables: absolute and relative.

Recall that we have previously studied extrema in functions of one variable, $y = f(x)$. In the context of two variable functions, $z = f(x, y)$, the definitions are similar.

Absolute Extrema of $f(x, y)$	**Relative Extrema of $f(x, y)$**
Absolute Maximum occurs at (x_0, y_0) if $f(x_0, y_0) \geq f(x, y)\ \forall(x, y)$ $\forall(x, y)$ in the domain D of f Absolute Minimum occurs at (x_0, y_0) if $f(x_0, y_0) \leq f(x, y)\ \forall(x, y)$ $\forall(x, y)$ in the domain D of f $\star$ Highest peak and lowest valley of an interval	If f is a function defined on a region containing (x_0, y_0), then: $f(x_0, y_0)$ is a relative maximum if $f(x, y) \leq f(x_0, y_0)\ \forall(x, y)$ in an open region containing (x_0, y_0) $f(x_0, y_0)$ is a relative minimum if $f(x, y) \geq f(x_0, y_0)\ \forall(x, y)$ in an open region containing (x_0, y_0) $\star$ All peaks and valleys of an interval



Why are extrema important? It is often helpful to know the largest and smallest values of a function of two variables. Calculating extrema can have benefits in areas of geography, physics, and economics. For examples, if we are looking to maximize profits or revenue, then we need to study the extrema of our function.

Finding Relative Extrema

First, we need to define **critical points** of a function f. A critical point of a function f defined on an open set D is a point (x_0, y_0) in D where either one of the following is true:

1. $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$
2. At least one of $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist

To determine the behavior of a critical point, we have the **Second Partials Test**. Suppose $f(x, y)$ has a critical point at $P_0(x_0, y_0)$ and f has continuous second-order partial derivatives in a disk centered at (x_0, y_0). The *discriminant* of f is $D = f_{xx}f_{yy} - f_{xy}^2$. Then we have the following results:

$D(a, b)$	$f_{xx}(a, b)$	**Type**
$+$	$-$	Rel. Max.
$+$	$+$	Rel. Min.
$-$	NA	Saddle Point
0	NA	Inconclusive

Example 1. Find all relative extrema and saddle points of the function $f(x,y) = x^3 + y^3 - 3xy + 6$

1. Find the critical points (find $f_x(x,y) = 0$ and $f_y(x,y) = 0$:

   $f_x = 3x^2 - 3y \quad \rightarrow \quad 3x^2 - 3y = 0 \qquad\qquad f_y = 3y^2 - 3x \quad \rightarrow \quad 3y^2 - 3x = 0$

   Now we have a system of nonlinear equations to solve. (If you need a refresher on solving systems of equations, check the resource under 1320: Algebra.)

   Solve for y: $\quad 3x^2 - 3y = 0 \quad \rightarrow \quad 3y = 3x^2 \quad \rightarrow \quad y = x^2$

   Substitute $y = x^2$ in the other equation: $\quad 3y^2 - 3x = 0 \quad \rightarrow \quad 3(x^2)^2 - 3x = 0$

   $$\rightarrow \quad 3x^4 - 3x = 0$$
   $$\rightarrow \quad 3x(x^3 - 1) = 0$$
   $$\rightarrow \quad x = 0 \text{ or } x = 1$$

   Since $y = x^2$, then $y = 0$ when $x = 0$ and $y = 1$ when $x = 1$

2. Classify critical points using the second partials test:

   $f_{xx} = 6x \quad f_{yy} = 6y \quad f_{xy} = -3 \quad \rightarrow \quad D = f_{xx}f_{yy} - f_{xy}^2 = (6x)(6y) - (-3)^2 = 36xy - 9$

   $D(0,0) = 36(0)(0) - 9 = -9 < 0 \quad \rightarrow \quad$ Using the table above, $(0,0)$ is a **saddle point**

   $D(1,1) = 36(1)(1) - 9 = 27 > 0 \quad \rightarrow \quad$ Using the table, we need to also check $f_{xx}(1,1)$

   $$f_{xx}(1,1) = 6(1) = 6 > 0$$

   $\therefore D(1,1) > 0$ and $f_{xx}(1,1) > 0$, $(1,1)$ is a **Rel. Min.**

Finding Absolute Extrema

Similar to the extreme value theorem of single variable functions, a function of two variables $f(x,y)$ attains both an absolute maximum and an absolute minimum on any closed, bounded set S where it is continuous. Here are steps to find the absolute extrema:

1. Find all critical points of f in S.
2. Find all points on the boundary of S where absolute extrema can occur (boundary points, critical points, endpoints, etc.).
3. Compute the value of $f(x_0, y_0)$ for each of the points (x_0, y_0) found in steps 1 and 2.
4. The absolute maximum of f on S is the largest of the values computed in step 3, and the absolute minimum is the smallest of the computed values.

Example 2. Find the absolute extrema of the function $f(x,y) = x^2 + xy + y^2 - 6x + 2$ on the rectangular plate $0 \leq x \leq 5$, $-3 \leq y \leq 0$.

1. **Find critical points:**

   $f_x = 2x + y - 6 \quad \rightarrow \quad 2x + y - 6 = 0 \qquad \text{and} \qquad f_y = x + 2y \quad \rightarrow \quad x + 2y = 0$

   Solve the system of equations:

   $x + 2y = 0 \quad \rightarrow \quad x = -2y$

   $2x + y - 6 = 0 \quad \rightarrow \quad 2(-2y) + y - 6 = 0$

   $\qquad\qquad\qquad\quad \rightarrow \quad -3y - 6 = 0$

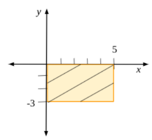
$\rightarrow \quad y = -2$

$\therefore$ When $y=-2$: $x=-2y \quad \rightarrow \quad x=-2(-2)=4$

Critical point: $(4,-2)$

2. **Find boundary points:**

   The rectangular plate $0 \le x \le 5$, $-3 \le y \le 0$ we are restricted to the area shown below:

   - Consider the right side boundary where $x=0$ and $-3 \le y \le 0$. Then:
     $g(y) = f(0,y) = 0^2 + 0y + y^2 - 6(0) + 2 = y^2 + 2$
     Now, let's check for critical points of $g(y) = y^2 + 2$:
     $g'(y) = 2y \quad \rightarrow \quad 2y = 0 \quad \rightarrow \quad y = 0$
     $\therefore$ We have the two **critical points** $(0,0),\ (0,-3)$.
   - Consider the left side boundary where $x=5$ and $-3 \le y \le 0$. Then:
     $g(y) = f(5,y) = 5^2 + 5y + y^2 - 6(5) + 2 = y^2 + 5y - 3$
     Now, let's check for critical points of $g(y) = y^2 + 5y - 3$:
     $g'(y) = 2y + 5 \quad \rightarrow \quad 2 + 5y = 0 \quad \rightarrow \quad y = \frac{-5}{2}$
     $\therefore$ We have the three **critical points** $(5,0),\ (5,-3),\ (5,\frac{-5}{2})$ since $\frac{-5}{2} \in [-3,0]$.
   - Consider the lower boundary where $y=-3$ and $0 \le x \le 5$. Then:
     $h(x) = f(x,-3) = x^2 + x(-3) + (-3)^2 - 6x + 2 = x^2 - 9x + 11$
     Now, let's check for critical points of $h(x) = x^2 - 9x + 11$:
     $h'(x) = 2x - 9 \quad \rightarrow \quad 2x - 9 = 0 \quad \rightarrow \quad x = \frac{9}{2}$
     $\therefore$ We have the **critical point** $(\frac{9}{2},-3)$ since $\frac{9}{2} \in [0,5]$.
   - Consider the upper boundary where $y=0$ and $0 \le x \le 5$. Then:
     $h(x) = f(x,0) = x^2 + x(0) + (0)^2 - 6x + 2 = x^2 - 6x + 2$
     Now, let's check for critical points of $h(x) = x^2 - 6x + 2$:
     $h'(x) = 2x - 6 \quad \rightarrow \quad 2x - 6 = 0 \quad \rightarrow \quad x = 3$
     $\therefore$ We have the **critical point** $(3,0)$ since $3 \in [0,5]$.

3. **Evaluate $f(x,y) = x^2 + xy + y^2 - 6x + 2$ for each critical point identified, then find largest and smallest values:**

   $f(4,-2) = (4)^2 + (4)(-2) + (-2)^2 - 6(4) + 2 = -10$ $\rightarrow$ Absolute Minimum of -10 at $(4,-2)$

   $f(0,0) = (0)^2 + (0)(0) + (0)^2 - 6(0) + 2 = 2$

   $f(0,-3) = (0)^2 + (0)(-3) + (-3)^2 - 6(0) + 2 = 11$ $\rightarrow$ Absolute Maximum of 11 at $(0,-3)$

   $f(5,0) = (5)^2 + (5)(0) + (0)^2 - 6(5) + 2 = -3$

   $f(5,-3) = (5)^2 + (5)(-3) + (-3)^2 - 6(5) + 2 = -9$

   $f(5,\frac{-5}{2}) = (5)^2 + (5)(\frac{-5}{2}) + (\frac{-5}{2})^2 - 6(5) + 2 = -9.25$

   $f(\frac{9}{2},-3) = (\frac{9}{2})^2 + (\frac{9}{2})(-3) + (-3)^2 - 6(\frac{9}{2}) + 2 = -9.25$

   $f(3,0) = (3)^2 + (3)(0) + (0)^2 - 6(3) + 2 = -7$

Practice Problems

1. Find all relative extrema and saddle points of the function $f(x, y) = 4xy - x^4 - y^4$

   [**Solution:** Saddle point at $(0, 0)$, Relative Maxima of 2 at $(1, 1)$ and 2 at $(-1, -1)$]

2. Find the absolute extrema of the function $f(x, y) = x^2 + xy + y^2 - 6x$ on the rectangular plate $0 \leq x \leq 5, \ \ -3 \leq y \leq 3$.

   [**Solution:** Absolute Maximum of 37 at $(5, -3)$ and Absolute Minimum of -12 at $(4, -2)$]